

# DEFAULT COMPENSATOR, INCOMPLETE INFORMATION, AND THE TERM STRUCTURE OF CREDIT SPREADS

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## Abstract

We provide a framework for the analysis of term structures of credit spreads on corporate bonds in the presence of informational asymmetries. While bond investors observe default incidents, we suppose that they have incomplete information on the firm's assets and/or the threshold asset level at which informed equity investors liquidate the firm. As a natural tool for the characterization of conditional default probabilities, prices of default-contingent claims, and credit spreads, we construct the compensator of default in terms of investors' threshold prior and the conditional running minimum asset distribution. With perfect asset observation, a new phenomenon appears: the default compensator is singular. Here an arrival intensity for default does not exist even though the default is completely unpredictable. In a setting where the assets of the firm follow a geometric Brownian motion, we show that the term structure of credit spreads is decreasing or hump-shaped, depending on the level of the current asset value. Spreads for maturities going to zero are only positive if the assets are at an historic low and the firm is quite risky. With imperfect asset observation, an arrival intensity for default does exist. This intensity is characterized through the compensator. In the geometric Brownian motion setting, the spread term structure is always decreasing with strictly positive spreads. **Key words:** incomplete information, credit spreads, compensator, intensity. **JEL Classification:** G12; G13

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# 1 Introduction

In the secondary bond market, it is typically difficult for investors to observe all parameters needed to assess the credit quality of an issuer. Investors are instead forced to estimate the financial health of an issuer based on the imperfect information which is publicly available. In this paper, we provide a new framework for studying the term structure of default risk and credit yield spreads on corporate bonds in such a situation.

The analysis of default-risky corporate debt is central to corporate finance, both from a theoretical and an empirical point of view. The so-called *structural* approach to model corporate default and corporate bond prices has its roots in the seminal work of Black & Scholes (1973); it has been fully developed by Merton (1974). These contributions shared the fundamental insight that corporate liabilities can be considered as contingent claims on the firm's assets. Subsequent research based on this contingent claims approach has aimed mainly at relaxing various restrictive assumptions concerning for example interest rates [e.g. Longstaff & Schwartz (1995)] and capital structure [Geske (1979)]. While in Merton (1974) the firm can default at debt maturity only, Black & Cox (1976) assumed that the firm may default at any time before the bond's maturity. This is described by defining the default event as the first time the firm's assets fall to some lower threshold level. This threshold can be imposed exogenously by bond safety covenants [Black & Cox (1976)] or endogenously by having the shareholders optimally liquidate the firm [e.g. Leland (1994), Leland & Toft (1996), and Anderson & Sundaresan (1996)].

In all these contributions public bond investors are assumed to be perfectly informed. But in practice there are informational asymmetries. While equity investors as firm insiders have complete information, for bond investors it is typically difficult if not impossible to directly observe the assets of an issuer and the threshold asset level at which equity investors liquidate the firm. Imperfect asset information is for instance due to (possibly intentional) noise and delays in accounting reports. Equity investors are likely to refrain from disclosing the chosen liquidation asset level to the public market in order to exploit this inside firm information. Their goal is to maximize the value of their own stake in the firm by extracting value from bond investors' stake.

Only recently Duffie & Lando (2001) addressed the issue of imperfect asset information. They supposed that investors know the firm's default threshold but cannot directly observe the issuer's assets. Instead, they receive noisy asset reports at discrete points in time. Duffie & Lando (2001) showed that in this case credit yield spreads are strictly positive and that an intensity process exists. Prior to default, the intensity can be interpreted as the conditional default arrival rate. They characterized this intensity in terms of the conditional asset density, given past accounting reports and survivorship. With perfect asset information however, investors can observe at any time the nearness of

the assets to the default threshold level. Consequently, if assets follow some continuous process they are warned in advance when a default is imminent. Therefore credit spreads go to zero with maturity going to zero and an intensity does not exist. Bond prices converge continuously to their default-contingent values – there is no surprise jump in prices upon default. These properties are empirically not plausible, cf. Sarig & Warga (1989) and Beneish & Press (1995).

The existence of an arrival intensity for default means that a structural model with imperfect asset observation is consistent with the so-called reduced-form or *intensity based* approach to corporate default. In this ad-hoc approach, the default occurs completely unexpectedly, by surprise so to speak. The default event is not causally modeled in terms of the firm's assets and liabilities, but is typically given exogenously. The stochastic structure of default is directly prescribed in terms of some arrival intensity process [e.g. Duffie & Singleton (1999), Duffie, Schroder & Skiadas (1996), Artzner & Delbaen (1995), Lando (1998), and Jarrow & Turnbull (1995), to mention a few]. Such an approach leads to empirically plausible results, as it implies surprise jumps in bond prices around default and strictly positive spreads. Defaultable security prices can be represented in terms of the intensity, leading to tractable valuation problems very similar to those arising in ordinary default-free term structure modeling.

In this paper, we present a new framework for studying the term structure of credit spreads with incomplete information on the issuer's default threshold level and/or assets in structural models. Our primary methodological contribution consists in considering the *compensator* of the default indicator process (the process that jumps from zero to one upon default) in its relation to bond prices and credit spreads. This compensator counteracts the jump in the default indicator such that the difference between compensator and indicator becomes a martingale. In that sense the compensator can be viewed as the fair fixed rate premium for a default insurance payment of one unit of account upon the default incidence. While the compensator always uniquely exists, the existence of an intensity as the density of the compensator is not always granted. In fact, we will provide a first example in which the compensator is singular and admits no intensity. In such situations the well-known intensity based representations of defaultable security prices and credit spreads break down.

We show that the properties of the compensator are intimately related to the extent of available information. With complete information the default is a predictable event and the compensator is trivial. We demonstrate that this implies that credit spreads go to zero with maturity going to zero, regardless of how the default is modeled. But with incomplete information the default is an unpredictable surprise event and the compensator is non-trivial. In that case conditional default probabilities and prices of default-contingent claims can be represented in terms of the compensator. Our price characterizations

are similar in spirit to those obtained in the reduced-form approach, but remain valid for a singular compensator which admits no intensity at all. This generalization of classic intensity based results will indeed be critical.

For general observation schemes with incomplete information on the issuer's default threshold level and/or assets, we construct the compensator. Based on this compensator, credit spreads can be calculated for any chosen asset observation scheme. We then distinguish two specific situations: perfect and imperfect asset observation. In case of *perfect asset observation*, the compensator is characterized in terms of investors' threshold prior and the running minimum asset value. Although the default occurs completely unpredictably, the *compensator is singular* and an arrival intensity for default does not exist at all. While hence a structural model based on incomplete default threshold observation is not consistent with a classic reduced-form representation of spreads, it is consistent with our compensator based representation. In a setting where the assets of the firm follow a geometric Brownian motion, we show that the spread term structure is decreasing or hump-shaped, depending on the level of the current asset value. But the short spread is only positive if the firm is quite risky, which is the case when its asset value is at an historic low. This suggests that incomplete threshold information loses its effect on credit spreads with increasing credit quality, since then the spread term structure approaches that with complete information.

The situation changes if, instead of or in addition to threshold uncertainty, investors have also *imperfect asset information*. In this case the compensator admits an intensity, meaning that in this case our structural model is consistent with a compensator based *and* a classic intensity based representation of spreads (both are equivalent here). The intensity is characterized through the compensator in terms of investors' threshold prior and the conditional distribution of the running minimum asset value, given all available asset information. This provides an alternative approach to Duffie & Lando's (2001) result, who calculate the intensity directly when the default threshold is observable. In a setting where the assets of the firm follow a geometric Brownian motion, we show that the spread term structure is always decreasing. Regardless of the riskiness of the firm, spreads are always strictly positive and short spreads are given by the intensity. This is empirically plausible.

The remainder of this paper is organized as follows. In Section 2, we present a structural default model with incomplete information. In Section 3, we work out the relation between the compensator, default time properties, default probabilities, default-contingent claim prices, and credit spreads. In Section 4, we characterize the default compensator. Settings with perfect and imperfect asset information are studied in some detail. In Section 5, the term structure of credit spreads is examined in a situation where investors' threshold prior is uniform and the issuer's assets follow a geometric Brownian motion. The Appendix contains all proofs.

## 2 Assets, Default, and Information

Uncertainty in the economy is modeled by a fixed probability space  $(\Omega, \mathcal{H}, P)$ , equipped with a filtration<sup>1</sup>  $(\mathcal{H}_t)_{t \geq 0}$  describing the information flow over time. Let us consider some given firm which has issued bonds on the financial market. We take as given some  $\mathbb{R}$ -valued stochastic process  $V = (V_t)_{t \geq 0}$ , where  $V_t$  is a sufficient statistic for the expected discounted future cash flows of the firm as seen from time  $t$ . We will therefore call  $V$  *asset process*.  $V$  is Markovian and continuous, and without loss of generality we normalize  $V_0 = 0$ . We denote by  $(\mathcal{F}_t)_{t \geq 0}$  the filtration generated by  $V$ . The *running minimum asset process*  $(M_t)_{t \geq 0}$  is defined by

$$M_t = \min\{V_s \mid 0 \leq s \leq t\}.$$

The firm is financed by equity and debt. Debt is modeled as a non-callable consol bond, paying coupons at some constant rate as long as the firm operates. When the firm stops servicing its contractual agreed obligations, we say it defaults. The firm then enters financial distress and some form of corporate reorganization takes place. Equityholders govern the firm and decide whether and when to stop debt service payments and to liquidate the firm. We suppose that shareholders choose to default if the firm's expected future cash flows are sufficiently low. The optimality of this policy has been verified by Duffie & Lando (2001). Thus we assume that there exists a random *default threshold*  $D \in \mathcal{H}_0$ , which is independent of  $(\mathcal{F}_t)$ , such that the shareholders liquidate the firm at the moment the asset value  $V$  falls for the first time to the threshold  $D < V_0 = 0$ . The firm's *default time*  $\tau$  is therefore given by

$$\tau = \inf\{t > 0 \mid V_t \leq D\}. \quad (1)$$

Let us denote the default indicator process by  $N = (N_t)_{t \geq 0}$  with  $N_t = 1_{\{t \geq \tau\}}$ . That is,  $N$  is zero before default and jumps to one upon default. We obtain immediately

$$\{\tau \leq t\} = \{M_t \leq D\}. \quad (2)$$

In our model, corporate claimants' access to inside firm information varies according to their role in governing the firm: equity and bond investors are *asymmetrically informed*. Shareholders govern the firm; they are assumed to have complete information about the firm's default threshold and assets. Their information flow is modeled by the filtration  $(\mathcal{H}_t)_{t \geq 0}$  generated by<sup>2</sup>

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(D). \quad (3)$$

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<sup>1</sup>Here and in the sequel, any filtration will be assumed to satisfy the usual conditions of right-continuity and completeness, see Brémaud (1980, III.5) for example.

<sup>2</sup>Here and in the sequel, we pass from the  $\sigma$ -algebras defined in (3) to the induced filtration  $(\mathcal{H}_t)$  without changing the notation.

Bond investors are outside investors and their access to inside firm information is limited. While bond investors observe the publicly announced default incident, the firm's default threshold cannot be directly observed in general. Equity investors aim at maximizing the value of their own stake in the firm by extracting value from bond investors' stake. Shareholders will therefore refrain from disclosing the default threshold value to the public bond market; they will rather use this inside firm information in their own interest. The issuer's assets are typically not traded publicly; as a result a direct observation of assets can be difficult for bond investors. We model the publicly available asset information by the filtration  $(\mathcal{A}_t)_{t \geq 0}$ . In the following examples we describe some realistic choices for this filtration.

**Example 2.1.** *Bond investors observe assets perfectly and we set*

$$\mathcal{A}_t = \mathcal{F}_t.$$

*This would be a reasonable assumption for a public firm, which allows to infer information on the firm's assets from the price of its shares.*

**Example 2.2.** *The issuers' assets may not be perfectly transparent to the secondary market. Duffie & Lando (2001) suggest that bond investors may instead receive at times  $t_1 < t_2 < \dots < t_m$  a noisy accounting report  $Y_{t_k} = V_{t_k} + U_{t_k}$ , where  $U_{t_k}$  is some independent noise random variable. We set*

$$\mathcal{A}_t = \sigma(Y_s, s \leq t, s \in \{t_1, \dots, t_m\}).$$

*The variance of  $U_{t_k}$  can be interpreted as a measure of the degree of accounting noise at time  $t$ . The  $U_{t_k}$  can be serially correlated, reflecting persistence of accounting noise in time, or correlated with the asset value  $V_{t_k}$ .*

**Example 2.3.** *Instead of receiving noisy asset reports at discrete points in time, bond investors may receive such reports continuously through time. Investors may be able to observe some auxiliary process  $Y$  whose drift  $\mu$  is modulated by the asset process  $V$  in that  $\mu = f(V_t, t)$  for some smooth function  $f$ , cf. Kusuoka (1999). We now set*

$$\mathcal{A}_t = \sigma(Y_s, s \leq t).$$

**Example 2.4.** *Bond investors have no asset information at all. In this case the filtration  $(\mathcal{A}_t)$  is trivial:  $\mathcal{A}_t = \{\Omega, \emptyset\}$  for all  $t \geq 0$ . Prior to the first noisy observation, this situation also holds in Example 2.2.*

Let us summarize. While public bond investors witness the default event, they cannot observe the issuer's default threshold and may have only incom-

plete information on the firm's assets. We thus model bond investors' information flow by a filtration  $(\mathcal{G}_t)_{t \geq 0}$  given by<sup>3</sup>

$$\mathcal{G}_t = \sigma(N_s, s \leq t) \vee \mathcal{A}_t. \quad (4)$$

Let us observe that the default time  $\tau$  is a  $(\mathcal{G}_t)$ -stopping time, meaning that the  $(0, \infty]$ -valued random variable  $\tau$  is such that for each time  $t \geq 0$  the event  $\{\tau \leq t\}$  is  $\mathcal{G}_t$ -measurable (loosely, at each time bond investors know whether a default has occurred or not).<sup>4</sup> In lack of default threshold information, public investors form a common *prior distribution*  $G$  on  $D$ , which we take as given. We assume that  $G$  is twice continuously differentiable with density  $g$ .

We maintain the following additional assumptions throughout. Shareholders are not permitted, say by insider legislation, to trade in the bond market. Otherwise shareholders could control the firm so as to maximize the value of their debt investments. Also, bond transactions could reveal inside firm information, for example on the true threshold or asset value. Finally, all agents are assumed to be risk-neutral. Hence we do not need to specify the market price of default risk for valuation purposes, which would be beyond the scope of this work. We refer to El Karoui & Martellini (2001) for an analysis of this issue in an (intensity based) equilibrium model with default.

### 3 Compensator, Prices, and Credit Spreads

In this section we will introduce the notion of the compensator of the default indicator process  $N$ . We will show that this compensator is the natural tool to study default probabilities, default-contingent claims, and credit yield spreads. We shall start by deriving general results that are then interpreted in the context of the model described in Section 2.

The probabilistic properties of the default stopping time  $\tau$  given by (1) will play a central role in the sequel. Let us therefore recall that  $\tau$  is called *predictable* if there is an increasing sequence of stopping times  $(T_n)$  such that  $\tau > T_n$  and  $\lim_n T_n = \tau$ . Intuitively, one can foretell the default event by observing a succession of 'forerunners'. We also say that  $(T_n)$  announces  $\tau$ . The

<sup>3</sup>Observe that  $\tau$  is not an  $(\mathcal{A}_t)$ -stopping time: having information  $\mathcal{A}_t$  allows not to deduce whether the default has occurred by time  $t$  or not. We therefore assume that  $(\mathcal{G}_t)$  is the progressive enlargement of  $(\mathcal{A}_t)$  with the random variable  $\tau$ . That is,  $(\mathcal{A}_t)$  is such that  $(\mathcal{G}_t)$  is the smallest filtration that includes  $(\mathcal{A}_t)$  such that  $\tau$  is a  $(\mathcal{G}_t)$ -stopping time.

<sup>4</sup>We remark that with respect to the filtration  $(\mathcal{H}_t)$ , the bond market is complete as soon as a riskless security and the assets of the firm are traded. But, due to the uncertainty regarding the threshold and/or assets, the bond market is incomplete with respect to the smaller filtration  $(\mathcal{G}_t)$ : there may be no perfect default hedge and the defaultable bond carries intrinsic risk. We refer to Föllmer & Schweizer (1990) for a characterization of hedging strategies which minimize the remaining risk in a general contingent claim model, in which incomplete information may lead to market incompleteness.

stopping time  $\tau$  is called *totally inaccessible* if  $P[\tau = T < \infty] = 0$  for all predictable times  $T$ . Here an announcing sequence does not exist. An inaccessible event is the probabilistic concept of a completely unpredictable phenomenon.

Noting that the default indicator process  $N$  is a submartingale, the Doob-Meyer decomposition theorem states that there exists a unique right-continuous, increasing, and predictable process  $A = (A_t)_{t \geq 0}$  with  $A_0 = 0$  and such that the difference process  $N - A$  is a  $(\mathcal{G}_t)$ -martingale. The process  $A$  is called the *compensator* of the one-jump point process  $N$  with respect to  $(\mathcal{G}_t)$ . The properties of the compensator are closely related to the probabilistic properties of the default time:  $A$  is continuous if and only if  $\tau$  is totally inaccessible. If  $A$  is absolutely continuous with respect to Lebesgue measure, i.e. if

$$A_t = \int_0^{t \wedge \tau} \lambda_s ds, \quad (5)$$

for some bounded progressively measurable process  $\lambda = (\lambda_t)_{t \geq 0}$ , then we say that  $\tau$  admits the *intensity*  $\lambda$ . Thus  $\tau$  must be an inaccessible stopping time if an intensity exists. On the other hand,  $\tau$  being inaccessible is not sufficient for an intensity to exist. If the intensity is predictable, it is essentially unique. By using the martingale property of the process  $N - A$ , it follows from (5) that on  $\{\tau > t\}$  the intensity satisfies

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} E[N_{t+h} - N_t | \mathcal{G}_t] = \lim_{h \downarrow 0} \frac{1}{h} P[\tau \in (t, t+h] | \mathcal{G}_t] \quad \text{a.s.} \quad (6)$$

Hence  $\lambda_t$  can be interpreted as the conditional event arrival rate at time  $t$ , given  $\mathcal{G}_t$  and that  $\tau > t$ . For more details we refer to Brémaud (1980, Ch. 2.3).

As a submartingale, the default indicator process  $N$  tends to rise on average. The idea of compensation via  $A$  involves the counteraction of this tendency in a predictable way such that the residual process  $N - A$  follows a martingale. Now consider a simple default insurance contract that stipulates a payment of one unit of account upon the default. Since  $E[dN_t] = E[dA_t]$ , the compensator can be viewed as the fair fixed rate premium for the contract. If an intensity exists, it can be thought of as the variable rate default insurance premium paid continuously through time up to the default event.

Let us emphasize that we do not assume the existence of a default arrival intensity in the sequel. This is essential as we shall see when we characterize the compensator in our model of Section 2. We are able to represent conditional default probabilities in terms of the compensator as follows.

**Proposition 3.1.** *Let the default stopping time  $\tau$  be totally inaccessible. If the process  $Y$  defined by  $Y_t = E[e^{A_t - A_\tau} | \mathcal{G}_t]$  is continuous at  $\tau$ , then for  $t < \tau$  conditional default probabilities are given by*

$$P[\tau \leq T | \mathcal{G}_t] = 1 - E[e^{A_t - A_\tau} | \mathcal{G}_t], \quad t \leq T.$$



Let us now consider the relation between the compensator and prices of default-contingent claims. A *default-contingent claim* is a security specified by a tuple  $(T, X)$ , which promises to pay at some fixed horizon  $T$  an amount given by the random variable  $X \in \mathcal{G}_T$ . If the security issuer defaults before  $T$  the security pays nothing; its payoff at  $T$  is thus given by  $X1_{\{\tau > T\}}$ . The canonical example of such a default-contingent claim is a defaultable zero-coupon bond maturing at  $T$ , for which  $X = 1$ . We take as given some adapted short rate process  $(r_t)_{t \geq 0}$  and assume that it is possible to invest in a locally riskless bank account with a value of  $\exp(\int_0^t r_s ds)$  at time  $t$ . We recall that agents are assumed to be risk-neutral. In Appendix B we consider the compensator under a change of probability measure, which is required in the valuation problem when we pass from the current risk-neutral world to a risk-averse one.

**Proposition 3.2.** *Let the default stopping time  $\tau$  be totally inaccessible. If the process  $Y$  defined by  $Y_t = E[Xe^{-\int_t^T r_s ds + A_t - A_T} | \mathcal{G}_t]$  is continuous at  $\tau$ , then the default-contingent claim  $(T, X)$  has at  $t < \tau$  a value of*

$$E[e^{-\int_t^T r_s ds} X 1_{\{\tau > T\}} | \mathcal{G}_t] = E[Xe^{-\int_t^T r_s ds + A_t - A_T} | \mathcal{G}_t], \quad t \leq T.$$

This price representation is remarkable because it does not involve the default time any more. Proposition 3.2 shows that the valuation of a default-contingent claim can be reduced to that of an ordinary contingent claim by simply adjusting discount factors for the prevailing default risk. Instead of discounting with respect to the locally riskless bank account with value  $\exp(\int_0^t r_s ds)$  at  $t$ , we discount with respect to the default-risky bank account having value  $\exp(\int_0^t r_s ds) \exp(\int_0^t dA_s) = \exp(\int_0^t r_s ds + A_t)$  at time  $t$ .

The problem of representing prices of default-contingent claims is not new. Elliott, Jeanblanc & Yor (2000) consider a similar problem in connection with the so called hazard process of  $\tau$ , which is equal to the compensator under some conditions. Bélanger, Shreve & Wong (2001) work in a closely related framework and provide a price representation which does not require the default to be unpredictable. Both papers distinguish between some initial filtration and the one enlarged by the default time. Other contributions focusing on a price representation with an inaccessible default always assume additionally the existence of a default intensity. While they require the compensator  $A$  of  $\tau$  to be absolutely continuous, we presume continuity only. If an intensity does in fact exist, we can write  $A_t = \int_0^{t \wedge \tau} \lambda_s ds$  and Proposition 3.2 leads under the no-jump condition to

$$E[e^{-\int_t^T r_s ds} X 1_{\{\tau > T\}} | \mathcal{G}_t] = E[Xe^{-\int_t^T (r_s + \lambda_s) ds} | \mathcal{G}_t], \quad t \leq T, \quad (7)$$

which is consistent with the results in the intensity-based literature, for example Lando (1998), Duffie & Singleton (1999), and Duffie et al. (1996) (the latter also relax the continuity condition). Hence in the valuation problem

the possibility of default requires an adjustment of interest rates: rather than discounting under the risk-free short rate  $r$ , one discounts under the default-adjusted rate  $R$  defined by  $R_t = r_t + \lambda_t$ .

Our next goal will be to relate the properties of the compensator to credit spreads on zero coupon bonds issued by the considered firm.<sup>5</sup> The *credit yield spread*  $S(t, T)$  is the difference between the yield at time  $t$  on a credit risky zero bond and that on a credit risk-free zero bond, both maturing at  $T$ . Assuming that defaults are independent of riskless rates, we have

$$S(t, T) = -\frac{1}{T-t} \ln P[\tau > T | \mathcal{G}_t], \quad t < T, \quad t < \tau. \quad (8)$$

The term structure of credit yield spreads at  $t$  is the schedule of  $S(t, T)$  against the horizon  $T$ . Of particular interest is the *short credit spread*, defined by

$$\lim_{T \downarrow t} S(t, T) = \frac{\partial}{\partial T} P[\tau \leq T | \mathcal{G}_t] \Big|_{T=t}, \quad t < \tau. \quad (9)$$

The short spread is the excess yield over the risk-free yield demanded by bond investors for assuming the default risk of the bond issuer over the infinitesimal time period  $(t, t + dt]$ . Its relation to the probabilistic properties of the default time is elaborated in the following result.

**Theorem 3.3.** *The following holds true for the default stopping time  $\tau$ .*

- (1) *If  $\tau$  is totally inaccessible then  $\lim_{T \downarrow t} S(t, T) \geq 0$  for  $t < \tau$ . Assume that  $\tau$  admits moreover a right-continuous and bounded intensity  $\lambda$  and that the process  $Y$  defined by  $Y_t = E[e^{-\int_t^T \lambda_s ds} | \mathcal{G}_t]$  is continuous at  $\tau$ . Then for  $t < \tau$  short spreads satisfy*

$$\lim_{T \downarrow t} S(t, T) = \lim_{T \downarrow t} \frac{1}{T-t} (A_T - A_t) = \lambda_t \quad a.s.$$

- (2) *Let  $\tau$  be predictable. Defining  $Z_n := \{k2^{-n} | k = 0, 1, \dots\}$  for  $n \geq 1$ , then for  $t < \tau$  short spreads satisfy*

$$\lim_{n \uparrow \infty} \sum_{t_i \in Z_n} S(t_i, t_{i+1}) 1_{\{t_i < t \leq t_{i+1}\}} = 0 \quad a.s.$$

Empirical studies indicate that a default is inaccessible rather than predictable. Sarig & Warga (1989) find that credit spreads remain in general

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<sup>5</sup>In our model described in Section 2, the capital structure of the firms is based on consol bonds having no fixed maturity and paying out a constant coupon to the bond investors. We can strip the consol coupon into a continuum of zero coupon bonds with recovery being pro-rata based on the default-free market value that the strips contribute to the consol. As for the analysis of the consol bond, it is therefore enough to consider zero bonds.

bounded away from zero. Also, we observe jumps in bond prices at or around the bankruptcy announcement, cf. Beneish & Press (1995) and Duffie, Pedersen & Singleton (2000), who consider sovereign bonds. If the default was a predictable event, prices would converge continuously to their default-contingent values; there would be no sudden drop in the value upon default.

Theorem 3.3 shows that the properties of the short spread derive *only* from the probabilistic properties of the default time. Since there is a one-to-one correspondence between default time and compensator properties, we can also say that the spread is determined by the compensator. How a default event is constructed plays a role only insofar as it determines the default time properties. This concerns in particular the asset process. By direct calculation, Duffie & Lando (2001) argue that short spreads are zero for a default being defined by first hitting of a Brownian motion to some constant boundary. Theorem 3.3 proves that this is in fact due to the default being predictable in this case. In any default model where the default time is predictable, the short spread is zero.

Theorem 3.3 has significant implications for the modeling of default, both in general and in particular for the model described in Section 2. First of all, on the basis of this result we can distinguish the essence of the two default-modeling paradigms, the structural and the intensity based approach. In the *intensity based approach*, one starts right away by assuming that the default time is totally inaccessible and admits some intensity  $\lambda$ . Short spreads are now positive by construction, since by Theorem 3.3  $\lambda$  constitutes the short spread. While this leads to empirical plausibility, the intensity based approach lacks economic intuition since the default time is typically taken as exogenously given; it is usually not explicitly specified why a firm defaults. In the *structural approach*, the default event is defined as the first time the firm's asset process hits some lower threshold. This makes sense from an economic point of view, because the asset process is a sufficient statistic for the firm's future cash flows. The resulting probabilistic properties of the default time and thus the short spread properties vary with the available information.

In the structural models presented in the literature the information between corporate claimants is typically symmetric: the secondary market has complete information. In our structural model proposed in Section 2, this would mean that bond investors can observe the issuer's assets, default threshold, and default event; their information at time  $t$  would be given by  $\mathcal{G}_t = \mathcal{H}_t$ . In this case investors would always be certain about the distance of the firm to default, i.e. the nearness of the asset value to the default threshold. Then, for shareholders and for bond investors alike, a default would not come as a surprise. There is in fact no separate 'timing risk' of default, since this timing risk is actually the risk of asset price changes. Indeed, given the continuity of the

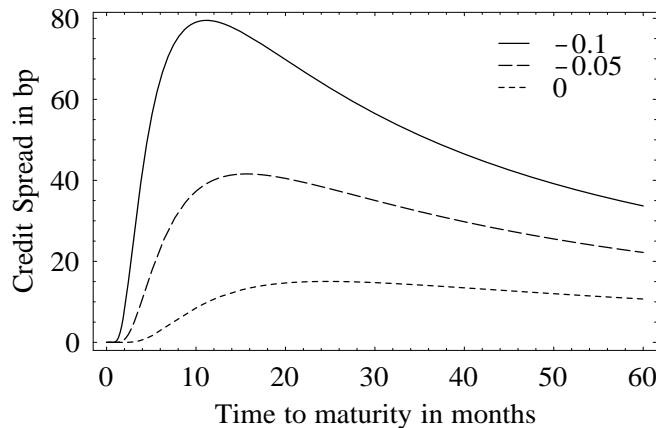


Figure 1: Term structure of credit spreads with completely informed bond investors, varying current asset value.

asset process, for all  $t \geq 0$  we would get

$$\{\tau(\omega) \leq t\} = \left\{ \inf_{0 \leq s \leq t} V_s(\omega) \leq D \right\} = \left\{ \lim_{n \uparrow \infty} \inf_{0 \leq s \leq t-n^{-1}} V_s(\omega) \leq D \right\} \in \mathcal{G}_{t-},$$

meaning that the default is predictable with respect to  $(\mathcal{G}_t)$ . This has three consequences. First, the market is complete and the default can be perfectly hedged given the asset value is a traded security. Second, since  $N$  is predictable, its decomposition is trivial. The compensator of  $N$  is  $A = N$  itself and an intensity does not exist. Proposition 3.2 does then not hold anymore. Third, by Theorem 3.3 credit spreads go to zero with maturity going to zero. Based on default probabilities given by

$$P[\tau \leq T | \mathcal{G}_t] = P[M_T \leq D | V_t] = P[M_{T-t} \leq D - V_t],$$

Figure 1 shows the term structure of credit spreads for varying current asset values in this situation (we set  $D = -0.3$ , all other parameters are those of Section 5). Indeed, regardless of the riskiness of the firm, spreads are zero for maturities up to approximately 2 months. In a structural model with complete information and a continuous asset process, investors do not demand a default risk premium on zero coupon debt whose maturity approaches zero. But this is not supported by empirical observations.

The probabilistic properties of the default time and thus the properties of the spread change if bond investors have only incomplete information on the characteristics of firms. Here investors are not able to observe the nearness of the assets to the default threshold, so that the default comes as a complete surprise event:  $\tau$  is totally inaccessible. The consequences are as follows. First, the default timing risk is not incorporated in asset price risk. That means

that even if assets are perfectly observed, the corporate bond market is incomplete. Second, by Theorem 3.3 short spreads are not necessarily zero as with complete information. Third, Proposition 3.2 provides a representation of default-contingent claim prices, irrespective of the existence of an intensity.

It appears that a structural approach based on incomplete information shares with an intensity based approach a fundamental property: the inaccessibility of defaults. This particular property leads on one hand to empirical plausibility with respect to bond price movements and credit spread term structures. On the other hand, by this property both approaches are integrated in the sense that they both admit a continuous default compensator. The two approaches are fully consistent as soon as there exists also an intensity in our structural model. This depends critically upon the extent of available information, as we will see in the next section.

## 4 Characterizing the Compensator

In the previous section we have shown that the default compensator characterizes credit spreads, and if it is continuous, default probabilities and prices of default-contingent claims. In this section we construct the compensator in terms of threshold prior and/or asset distribution for our structural default model of Section 2. Though the (continuous) compensator exists in any case, we will see that imperfect default threshold observation alone is not sufficient for an intensity to exist. In those cases where an intensity does exist, it is characterized through the compensator.

### 4.1 General Incomplete Information Case

In this section we consider the default compensator for general situations of incomplete information, where bond investors information filtration is given by  $\mathcal{G}_t = \sigma(N_s, s \leq t) \vee \mathcal{A}_t$ , cf. (4). While default incidents are observable, thresholds are unknown and the asset information  $\mathcal{A}_t$  may be complete or incomplete. In this case defaults are completely unpredictable and the default compensator is non-trivial.

Let us introduce the conditional survival probability  $L_t$  for time  $t$ , given all available asset information at  $t$ ,

$$L_t = P[\tau > t \mid \mathcal{A}_t]. \quad (10)$$

The process  $L = (L_t)_{t \geq 0}$  is called the *survival process*. Note that  $L_0 = 1$  by (1) and assume that  $L_t > 0$  for all  $t > 0$ . We let  $L_{t-} = \lim_{s \uparrow t} L_s$  and set  $L_{0-} = 1$ . We observe that  $L$  is a supermartingale; according to the Doob-Meyer decomposition theorem there is a unique increasing  $(\mathcal{A}_t)$ -predictable process  $K$  such that  $K + L$  is a  $(\mathcal{A}_t)$ -martingale.  $K$  is called the compensator of  $L$ ,

not to be confused with the default compensator  $A$  of the default indicator process  $N$ . As in Yor (1994), Elliott et al. (2000), and Jeanblanc & Rutkowski (1999), the latter can be represented in terms of the former.

**Theorem 4.1.** *Let  $K$  denote the  $(\mathcal{A}_t)$ -compensator of  $L$ . The  $(\mathcal{G}_t)$ -default compensator  $A$  is given by*

$$A_t = \int_0^{t \wedge \tau} \frac{dK_s}{L_{s-}}, \quad t \geq 0.$$

*If  $L$  is decreasing and continuous, then we have in particular  $A_t = -\ln L_{t \wedge \tau}$ .*

The characterization of the default compensator provided above holds for general situations of incomplete information on thresholds and assets. Given specific assumptions on investors' information structure, such as those made in Examples 2.1 – 2.4, more explicit representations can be obtained. In the following subsections we shall vary the structure of the filtration  $(\mathcal{A}_t)$ , where we distinguish in particular between situations with perfect and imperfect asset observation.

## 4.2 Perfect Asset Observation

Let us assume that the issuer's assets are perfectly observable, while the default threshold is not (Example 2.1). The auxiliary filtration  $(\mathcal{A}_t)$  coincides in this case with the asset filtration:  $\mathcal{A}_t = \mathcal{F}_t$ . Noting that the threshold  $D$  is independent of assets, the survival process  $L$  can be written as

$$L_t = P[\tau > t \mid \mathcal{F}_t] = P[D < M_t \mid \mathcal{F}_t] = G(M_t),$$

where  $G$  is the (given) prior distribution of  $D$ . The running minimum asset process  $M$  is continuous and decreasing, and so is the survival process  $L$ . In view of Theorem 4.1, we have thus shown the following.

**Theorem 4.2.** *If the issuer's assets are perfectly observable but its default threshold is not, then the  $(\mathcal{G}_t)$ -default compensator  $A$  is given by*

$$A_t = -\ln G(M_{t \wedge \tau}), \quad t \geq 0.$$

Using the fact that the quadratic variation of  $M$  is zero, by Itô's formula we can obtain the following alternative compensator characterization:

$$A_t = -\int_0^{t \wedge \tau} \frac{g(M_s)}{G(M_s)} dM_s, \quad t \geq 0,$$

where  $g$  is the prior density of  $D$  (this can as well be derived from the first part of Theorem 4.1 by noting that  $K = 1 - L$  and hence  $dK_t = -g(M_t)dM_t$ ).

If the default threshold is not observable by public bond investors, they can calculate the compensator in terms of their threshold prior and the observable running minimum asset value. The asset distribution does not enter the characterization of  $A$ , i.e. the compensator does not depend on the choice of the underlying asset process.<sup>6</sup> Default probabilities and prices of default-contingent claims can then be characterized in terms of this continuous compensator, as shown in Propositions 3.1 and 3.2. The continuity of the compensator provides a formal proof that with incomplete threshold observation the default is in fact a completely unpredictable surprise event for the bond investors.

But what can we say about the default arrival intensity? Because the Lebesgue measure of the set

$$\{t \geq 0 : V_t = M_t\}$$

is zero, the measure induced by the continuous process  $M$  is not absolutely continuous with respect to Lebesgue measure. It follows that the default compensator is not absolutely continuous either. Although the default is an unpredictable event, an arrival intensity process for default does not exist. That means that a causal structural default model in which the default threshold is nonobservable is not consistent with an intensity based approach to default, in which prices of defaultable claims can be represented in terms of an intensity process.

Now consider the implied credit spread term structure properties. By Theorem 3.3 (1), the spread  $S(t, T)$  remains non-negative for  $T \rightarrow t$ . But we can say even more in view of our compensator characterization. On the set  $\{V_t = M_t\}$ , when the asset value is at a 'historic low', we have  $S(t, T) > 0$  for  $T \rightarrow t$ . On the other hand, on  $\{V_t > M_t\}$ , we have  $S(t, T) = 0$  for  $T \rightarrow t$ , despite the fact that  $\tau$  is inaccessible. This means that, depending on the firm's current asset value, the credit spread term structure with incomplete information as in Example 2.1 can be very similar to the one which appears in the case of complete information. This property corresponds to the fact that, whenever assets  $V_t > M_t$  and  $\tau > t$ , bondholders are, loosely spoken, in a relieved position which is similar to that with complete information. Having incomplete information  $\mathcal{G}_t$  allows to deduce that  $D < M_t$ ; hence the firm cannot default immediately if the assets  $V$  follow a continuous process. This results in zero short term default probabilities and thus zero spreads. If in contrast  $V_t = M_t$ , then the firm can default immediately and short term default probabilities are strictly positive.

Credit spreads with perfect asset observation are further studied in Section 5. We will show that, depending on the current asset value, incomplete

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<sup>6</sup>In the model of Duffie & Lando (2001) where the default threshold is observable but assets are not, the opposite holds: the compensator that would be obtained is determined by the asset process only, cf. (13) below.

threshold information can lead to *decreasing* and *hump shaped* term structures of credit spreads.

### 4.3 Imperfect Asset Observation

We now study the default compensator when issuer's assets can only be imperfectly observed (Examples 2.2 – 2.4). We continue to assume that the default threshold is not observable; our results cover then also situations where the threshold is in fact known. While with perfect asset information a default arrival intensity does not exist, with imperfect asset observation an intensity can be established under technical conditions.

For  $t > 0$ , we denote by  $H(t, \cdot, \omega)$  the regular conditional distribution function of the running minimum asset value  $M_t$ , given the imperfect asset information  $\mathcal{A}_t$ . We can then write for the survival process

$$L_t = P[\tau > t \mid \mathcal{A}_t] = E[1 - H(t, D)] = 1 - \int_{-\infty}^0 H(t, x) g(x) dx. \quad (11)$$

Now Theorem 4.1 leads immediately to the following result.

**Proposition 4.3.** *Assume that bond investors cannot observe the issuer's default threshold and have imperfect asset information. If the conditional distribution  $H(t, \cdot)$  is continuous and increasing in  $t$  on  $(0, \infty)$ , then the  $(\mathcal{G}_t)$ -default compensator  $A$  is given by*

$$A_t = -\ln \left( 1 - \int_{-\infty}^0 H(t \wedge \tau, x) g(x) dx \right), \quad t \geq 0.$$

In case assets cannot be observed at all and investors have only survivorship information (Example 2.4), the filtration  $(\mathcal{A}_t)$  is trivial. Now  $H$  is a deterministic continuous and increasing function of time, and so  $L$  satisfies the conditions of Proposition 4.3. This case is further studied in Section 5 under explicit assumptions on the distribution of threshold and assets.

Let us now consider the intensity with imperfect asset observation.

**Theorem 4.4.** *Assume that bond investors cannot observe the issuer's default threshold and have imperfect asset information. For  $t > 0$  and almost every  $\omega$ , let  $|h^{-1}E[H(t+h, x) - H(t, x) \mid \mathcal{A}_t](\omega)|$  have an upper bound which is integrable with respect to  $g(x)dx$ . Assume furthermore that  $h^{-1}E[H(t+h, x) - H(t, x) \mid \mathcal{A}_t] \rightarrow \dot{H}(t, x) := \frac{\partial}{\partial t}H(t, x)$ . For all  $x \leq 0$  and all  $t > 0$ , let  $H(t, x, \omega)$  and  $\dot{H}(t, x, \omega)$  be bounded for almost every  $\omega$ . Let, for all  $x \leq 0$ , the processes  $(H(t, x))_{t \geq 0}$  and  $(\dot{H}(t, x))_{t \geq 0}$  be  $(\mathcal{G}_t)$ -progressively measurable. Then the  $(\mathcal{G}_t)$ -default compensator admits an intensity  $\lambda$  and is given by*

$$A_t = \int_0^{t \wedge \tau} \lambda_s ds \quad \text{with} \quad \lambda_t = \frac{\int_{-\infty}^0 \dot{H}(t, x) g(x) dx}{1 - \int_{-\infty}^0 H(t, x) g(x) dx}, \quad t > 0.$$



In a situation with nonobservable default threshold and imperfect asset observation, the default arrival intensity can be calculated in terms of investors' threshold prior and the conditional running minimum asset value distribution, given the imperfect asset information available to the market.<sup>7</sup> What happens if  $D \in \mathcal{G}_0$  and the threshold is a priori known to the bond investors? Under the conditions of Theorem 4.4, an intensity does still exist:

$$\lambda_t = \frac{\dot{H}(t, D)}{1 - H(t, D)}, \quad t \in (0, \tau).$$

A causal structural model with incomplete asset information, regardless of whether the default threshold is observable or not, is therefore consistent with an intensity based approach to default, where defaultable security prices can be represented in terms of an intensity process, cf. (7). Equivalently, default probabilities and prices can be represented in terms of the corresponding compensator, cf. Propositions 3.1 and 3.2. Theorem 4.4 links investors' beliefs, fundamental firm variables, and arrival intensity. In that sense, a structural model based on imperfect asset observation provides an economic underpinning for the ad-hoc intensity based models in the literature, where intensities are typically given exogenously.

In order to clarify the structure of the intensity, in the following we shall provide an equivalent characterization of the compensator.

**Proposition 4.5.** *Suppose that bond investors cannot observe the issuer's default threshold and have imperfect asset information. Assume that  $H$  satisfies the conditions of Theorem 4.4 and define for  $x \leq 0$  and  $t > 0$*

$$m_t(x) := \lim_{h \downarrow 0} \frac{\frac{1}{h} E[H(t+h, x) - H(t, x) | \mathcal{A}_t]}{1 - H(t, x)} = \frac{\dot{H}(t, x)}{1 - H(t, x)}.$$

*For  $t < \tau$ , suppose that  $D$  admits a conditional density  $g_t(\cdot, \omega)$  given  $\mathcal{G}_t$ . Assume that for all fixed  $x \leq 0$  and all  $t$ ,  $g_t(x, \omega)$  is bounded for almost every  $\omega$ . Also, for all fixed  $x \leq 0$ , suppose that the process  $(g_t(x))_{t \geq 0}$  is  $(\mathcal{G}_t)$ -progressively measurable. Then the  $(\mathcal{G}_t)$ -default compensator  $A$  admits an intensity  $\lambda$  given by*

$$A_t = \int_0^{t \wedge \tau} \lambda_s ds \quad \text{with} \quad \lambda_t = \int_{-\infty}^0 m_t(x) g_t(x) dx, \quad t > 0.$$

In the vast majority of structural approaches to default, the assets of the issuer are traditionally modeled as a Brownian motion. For this specific distributional choice we now identify the processes  $(m_t(x))_{t \geq 0}$  introduced in the previous result. This allows to immediately apply our results to the particular

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<sup>7</sup>Note that the intensity is  $(\mathcal{A}_t)$ -measurable.

settings appearing in the structural literature. We show that at any time  $t$  before default,  $m_t(x)$  can be expressed in terms of the asset volatility and the conditional density of the asset value  $V_t$ , given the bond market information  $\mathcal{G}_t$  and that  $D = x$ .

**Proposition 4.6.** *Assume that bond investors cannot observe the issuer's default threshold and have imperfect asset information. Suppose that the firm's asset value follows a Brownian motion with drift  $\mu \in \mathbb{R}$  and volatility  $\sigma > 0$ ,*

$$dV_t = \mu dt + \sigma dB_t, \quad (12)$$

where  $B$  is a standard Brownian motion. For  $t < \tau$ , suppose furthermore that  $V_t$  admits a conditional density  $a(t, x, \cdot, \omega)$  given  $\mathcal{G}_t$  and  $D := x$  with support  $[x, \infty)$ . We assume that, for each  $(t, x, \omega)$ ,  $a(t, x, \cdot, \omega) = 0$  on  $(-\infty, x]$  and  $a(t, x, \cdot, \omega)$  is continuously differentiable on  $(x, \infty)$  and differentiable from the right at  $x$ . Also, for almost every  $\omega$ , the derivative  $|a_z(s, x, z, \omega)|$  is bounded on sets of the form  $\{(s, x, z) : 0 \leq s \leq t, -\infty < x < 0, x \leq z < \infty\}$ . Then

$$m_t(x) = \frac{1}{2} \sigma^2 a_z(t, x, x), \quad t \geq 0.$$

The asset density can be computed explicitly in some cases. Consider the information structure described in Example 2.2 and assume that the firm's asset value  $V$  satisfies (12). Suppose that bond investors receive a noisy asset report  $Y_t = V_t + U$  at time  $t$ , where  $U$  is a noise variable independent of  $V$  and  $D$ , with given density  $q$ . First note that

$$P[M_t \in dy, V_t \in dz | Y_t] = \frac{q(Y_t - z) \varphi(t, y, z) dy dz}{p(t, Y_t)},$$

where  $\varphi(t, y, z)$  is the joint density of  $(M_t, V_t)$ , which is available explicitly, cf. Borodin & Salminen (1996).  $p(t, \cdot)$  is the density of  $Y_t$ , which can be obtained via convolution of  $q$  and density of  $V_t$ , given by  $\int_{-\infty}^0 \varphi(t, y, \cdot) dy$  (explicit as well, cf. Borodin & Salminen (1996)). For  $t < \tau$  we get by Bayes' rule

$$\begin{aligned} a(t, x, z) dz &= P[V_t \in dz | Y_t, M_t > D := x] = \frac{P[M_t > x, V_t \in dz | Y_t]}{P[M_t > x | Y_t]} \\ &= \frac{q(Y_t - z) \int_x^0 \varphi(t, y, z) dy dz}{\int_x^0 \int_x^\infty q(Y_t - v) \varphi(t, y, v) dv dy}. \end{aligned}$$

Now consider the information structure described in Example 2.4 and assume again that the firm's asset value  $V$  satisfies (12). We get

$$a(t, x, z) dz = P[V_t \in dz | M_t > D := x] = \frac{\int_x^0 \varphi(t, y, z) dy dz}{1 - H(t, x)},$$

which is also available explicitly, cf. Borodin & Salminen (1996).

#### 4.4 The Intensity Result of Duffie & Lando

Recently Duffie & Lando (2001) established the existence of a default intensity in a structural model with incomplete information on the firm's assets only. They assumed that assets  $V$  satisfy (12) and that the default time is given by  $\tau = \inf\{t : V_t \leq d\}$  for some *constant*  $d$ . Bond investors receive noisy asset reports at discrete dates  $t_k$ , cf. Example 2.2. Duffie & Lando (2001) established an intensity  $\lambda$  for  $\tau$  via (6) and showed that<sup>8</sup>

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} P[\tau \in (t, t+h] | \mathcal{G}_t] = \frac{1}{2} \sigma^2 f_x^d(t, d), \quad t \in (0, \tau), \quad (13)$$

where  $f^d(t, \cdot)$  is the conditional density of  $V_t$  given  $\mathcal{G}_t$  and survivorship. It is easily seen that this result follows as well from our Proposition 4.6 by assuming that the default threshold  $D$  is a priori known to the bond investors,  $D \in \mathcal{G}_0$ . We then obtain for the intensity  $\lambda_t = \frac{1}{2} \sigma^2 a_z(t, D, D)$ . Both results coincide as soon as  $a(t, x, z) = f^x(t, z)$  for all  $t < \tau$  and  $z \in [x, \infty)$ . For a single noisy observation  $Y_t = V_t + U$ , Duffie & Lando (2001) calculate the density  $f^x(t, z)$  for the a priori observable threshold  $D := x$  as follows:

$$f^x(t, z) dz = \frac{q(Y_t - z) \pi(-x, z - x, t) P[V_t \in dz]}{\int_x^\infty q(Y_t - v) \pi(-x, v - x, t) P[V_t \in dv]},$$

where, taking  $V_t$  to be a Brownian bridge with  $V_0 = x$  and  $V_t = y$ ,  $\pi(x, y, t)$  is the probability that  $\min\{V_s : 0 \leq s \leq t\} > 0$ . The equivalence between their asset density characterization and ours is obvious:

$$\begin{aligned} \pi(-x, z - x, t) P[V_t \in dz] &= P[\min_{0 \leq s \leq t} V_s > x | V_0 = 0, V_t = z] P[V_t \in dz] \\ &= P[M_t > x, V_t \in dz] \\ &= \int_x^0 \varphi(t, y, z) dy dz, \end{aligned}$$

and thus  $a(t, x, z) = f^x(t, z)$ . Assuming normality of  $U$ , Duffie & Lando (2001) compute the asset density explicitly in terms of the normal distribution function. Using this result, we can directly verify that  $a(t, x, \cdot)$  indeed satisfies the conditions of Proposition 4.6.

Let us emphasize the difference in the applied methodologies. We characterize the always existing default compensator, which in turn characterizes the default arrival intensity, if it exists. Duffie & Lando (2001), in contrast,

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<sup>8</sup>Using the fact that the intensity does not depend on the asset's drift, Duffie & Lando (2001) extend to the case where the asset value solves the SDE  $dV_t = \mu(V_t, t)dt + \sigma(V_t, t)dW_t$  for  $\mu$  and  $\sigma$  satisfying technical conditions. Then the intensity is for  $t \in (0, \tau)$  given by  $\lambda_t = \frac{1}{2} \sigma^2(d, t) f_x^d(t, d)$ .

calculate the intensity directly as the limit (6). If an intensity does in fact exist, both approaches lead to equivalent results. The existence of an intensity is however not granted in general. Here the generality of the compensator-based framework pays off: the compensator exists in any case and is represented by Theorem 4.1, *irrespective* of the existence of an intensity.

## 5 Empirical Implications

In this section we illustrate the significant empirical implications of incomplete information on the term structure of credit spreads. We specialize in the general setup of Section 2 by placing explicit assumptions on prior threshold distribution, asset distribution, and information structure.

We assume that the total market value  $Z$  of the firm follows a geometric Brownian motion with constant drift  $m \in \mathbb{R}$  and volatility  $\sigma > 0$ . That is,  $Z_t = Z_0 e^{V_t}$  with initial value  $Z_0 > 0$ . Here  $V$  is a Brownian motion with drift  $\mu = m - \frac{1}{2}\sigma^2$ , i.e.,  $V_t = \mu t + \sigma B_t$  with  $B$  being a standard Brownian motion. In the sequel we take  $V$  to be the 'asset process' in the sense of Section 2.

We assume furthermore that the a-priori default threshold distribution with respect to  $Z$  is uniform on  $(0, Z_0)$ . This choice corresponds to uninformed investors not having any specific knowledge on the default barrier. That implies that the a-priori distribution of  $D$  with respect to  $V$  is represented by

$$G(x) = g(x) = e^x, \quad x \in (-\infty, 0). \quad (14)$$

### 5.1 Credit Spreads With Perfect Asset Observation

We start by assuming that the issuer's assets are perfectly observable by bond investors, while the default threshold is unknown (Example 2.1). Given this information structure and (14), Theorem 4.2 implies that the continuous default compensator  $A$  is simply given by the asset's running minimum:

$$A_t = -M_{t \wedge \tau}.$$

Clearly  $M_t$  is not absolutely continuous and hence a default intensity does not exist. According to Proposition 3.1, the conditional default probability at time  $t < \tau$  for the horizon  $T > t$  is then

$$\begin{aligned} P[\tau \leq T \mid \mathcal{G}_t] &= 1 - e^{-M_t} E[e^{M_T} \mid \mathcal{F}_t, M_t > D] \\ &= 1 - \int_{-\infty}^{M_t} e^{x-M_t} dP[M_{T-t} \leq x - V_t] \\ &= \int_{-\infty}^{M_t} e^{x-M_t} \Psi(T-t, x - V_t) dx, \end{aligned}$$

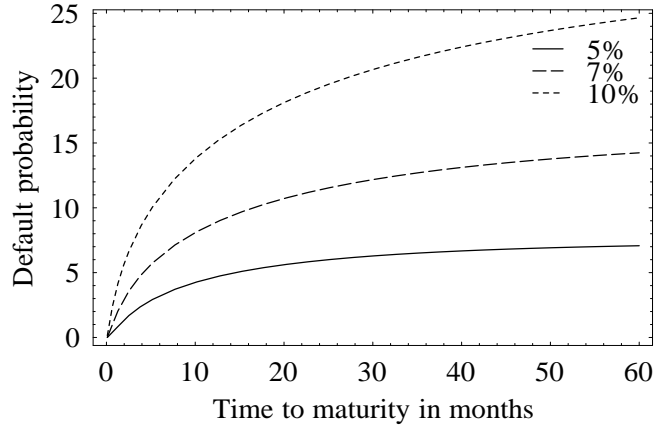


Figure 2: Term structure of conditional default probabilities, varying asset volatility.

where we have used the (strong) Markov property of Brownian motion in the second line. By  $\Psi(t, \cdot)$  we denote the unconditional distribution function of  $M_t$ ,

$$\Psi(t, x) := 1 - \Phi\left(\frac{\mu t - x}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2\mu x}{\sigma^2}\right) \Phi\left(\frac{x + \mu t}{\sigma\sqrt{t}}\right), \quad (15)$$

for  $x \leq 0$  and  $t > 0$ .  $\Phi$  is the standard normal distribution function. Since  $\Psi(t, \cdot)$  is continuous in  $t$ , the no-jump condition of Proposition 3.1 is satisfied. Figure 2 plots the term structure of conditional default probabilities for varying asset volatilities. As expected, default probabilities are increasing in asset volatility, or business risk. Unless noted otherwise, our base case parameter set is as follows: asset drift  $\mu = 1\%$ , asset volatility  $\sigma = 5\%$ , running minimum asset level  $M_t = -0.1$ , and asset level  $V_t = -0.1$ .

Assuming zero recovery and that the default is independent of risk-free interest rates, from (8) we have for the credit yield spread

$$S(t, T) = -\frac{1}{T-t} \ln \left( 1 - \int_{-\infty}^{M_t} e^{x-M_t} \Psi(T-t, x-V_t) dx \right), \quad t < T, \quad t < \tau.$$

In Figure 3 we graph the term structure of credit spreads for varying current asset levels. Depending on the current asset value, two distinct term structure shapes appear. If the asset value  $V_t$  is currently at its historic low  $M_t$ , then the term structure is decreasing and the spread is strictly positive for maturities near zero. Compare this to the case of complete information displayed in Figure 1. The downward-sloping shape of the term structure can be intuitively explained as follows. Since assets 'test' its historic lows, at time  $t$  the bond is fairly risky. In the short run negative shocks on the asset value can quickly

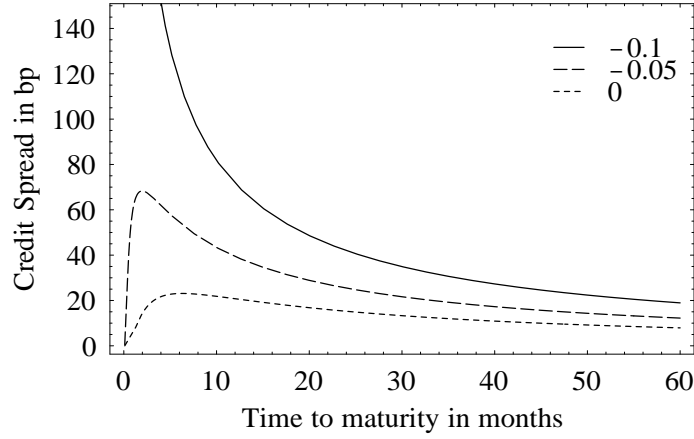


Figure 3: Term structure of credit spreads, varying current asset value.

lead to a default before time  $T > t$ . Due to the positive drift in the firm value, there is however room for improvement over time and only less potential to worsen. As a result, the spread decreases with the horizon  $T$ .

As soon as the current asset value  $V_t$  is increased above the level of the minimum asset value to date  $M_t$ , we witness a downward shift in the spread curve towards a hump shaped term structure. That is, in case  $V_t > M_t$  the term structure shape with incomplete information approaches that with complete information, where short spreads are zero. Clearly, as the (unknown) default threshold  $D$  must be below  $M_t$  if the firm still operates, if the firm value cannot jump higher firm values correspond to zero short-term default probabilities and thus zero short spreads. There remains however a difference for intermediate maturities of up to approximately 2 months, compare Figures 1 and 3 for these maturities. In essence, the effect of incomplete threshold observation becomes less important for firms with somewhat improved credit quality (firms whose assets are traded over their historic lows). For high quality firms with significant assets incomplete information hardly matters.

As Sarig & Warga (1989) report, a hump shaped term structure is typically observed for junk quality. The shift of the term structure from hump shaped to monotone decreasing can be considered as an increase in risk in the short run, as the assets deteriorate towards their historic low. In this sense the decreasing pattern is a characteristic for very high risk junk quality.

The spread term structure properties implied by this variant of our model seem very similar to those obtained by Merton (1974) in his pioneering work based on complete information. Merton finds that if the risklessly discounted face value of the zero bonds financing a firm besides equity is larger than the firm value, the term structure is decreasing; otherwise it is hump shaped.

Thus the decreasing term structure corresponds to a very risky firm, as in our setting.

In the model of Duffie & Lando (2001), where only the issuer's assets are imperfectly observed, the spread term structure appears to be hump shaped with strictly positive short spreads. With unknown default threshold but complete asset information, the term structure can be hump shaped or decreasing, depending on current assets. Although the default time is unpredictable, short spreads are only strictly positive if the asset value is at its running minimum. This shows that incomplete information does only in certain cases lead to empirically observed spread properties.

## 5.2 Credit Spreads With No Asset Observation

Let us now assume that bond investors have no information on the issuer's default threshold and assets at all and only survivorship information is available (Example 2.4). In this case the auxiliary filtration  $(\mathcal{A}_t)$  is trivial and the  $\mathcal{A}_t$ -conditional distribution function  $H(t, \cdot)$  of  $M_t$  is given by  $H(t, x) = \Psi(t, x)$  for  $\Psi$  defined in (15). According to Proposition 4.3, the default compensator is then

$$A_t = -\ln \left( 1 - \int_{-\infty}^0 \Psi(t \wedge \tau, x) e^x dx \right). \quad (16)$$

Letting  $\phi$  denote the standard normal density function, the derivative  $\dot{H}$  of  $H$  with respect to  $t$  is given by  $\dot{H}(t, x) = \dot{\Psi}(t, x)$  with

$$\begin{aligned} \dot{\Psi}(t, x) = & \frac{1}{2\sigma} \left[ \left( \frac{\mu}{\sqrt{t}} - \frac{x}{\sqrt{t^3}} \right) \exp \left( \frac{2\mu x}{\sigma^2} \right) \phi \left( \frac{x + \mu t}{\sigma \sqrt{t}} \right) \right. \\ & \left. - \left( \frac{x}{\sqrt{t^3}} + \frac{\mu}{\sqrt{t}} \right) \phi \left( \frac{\mu t - x}{\sigma \sqrt{t}} \right) \right], \quad x \leq 0, \quad t > 0, \end{aligned}$$

so that we immediately obtain for the default intensity

$$\lambda_t = \frac{\int_{-\infty}^0 \dot{\Psi}(t, x) e^x dx}{1 - \int_{-\infty}^0 \Psi(t, x) e^x dx}, \quad t \in (0, \tau), \quad (17)$$

as expected in view of Theorem 4.4. The intensity is a deterministic continuous function of time, asset drift and asset volatility only.  $\lambda$  is therefore unique, cf. Brémaud (1980, Theorem II T12). Figure 4 graphs  $\lambda$  as a function of time; this depicts the profile of short spreads over time, cf. Theorem 3.3. In line with intuition, the default intensity is increasing in the degree of business risk, as proxied by asset volatility  $\sigma$ . With a positive asset value drift  $\mu$ , the intensity is decreasing in time: conditional on survivorship, the local probability of hitting

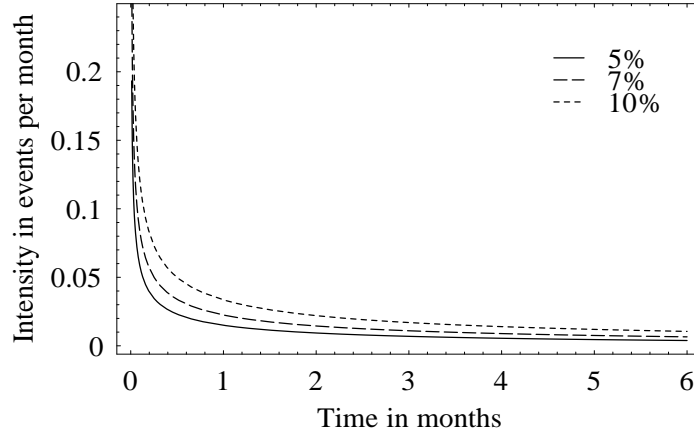


Figure 4: Default arrival intensity, varying asset volatility.

the default threshold decreases with the passage of time as assets increase on average.

For comparison, let us also consider the case when the issuer's default threshold  $D$  is observable,  $D \in \mathcal{G}_0$ . From (17) we then obtain explicitly

$$\lambda_t = \frac{\dot{\Psi}(t, D)}{1 - \Psi(t, D)}, \quad t \in (0, \tau). \quad (18)$$

Figure 5 shows the intensity in this case for a varying asset volatility (we set  $D = -0.1$ ). In contrast to the decreasing pattern with nonobservable threshold, here the intensity first sharply increases and then decreases with time (it is hump-shaped). Up to a certain point in time, assets are likely to fall from  $V_0 = 0$  to  $D$  and the local probability of hitting the threshold  $D$ , conditional on survivorship, increases. With a positive asset drift, after that certain point in time it is fairly unlikely that assets are close to  $D$ , given survivorship. It is more likely that assets move further away from  $D$ , thereby leading to decreasing intensities of default.

There are now several equivalent ways to calculate default probabilities. With the deterministic continuous default compensator (16), by Proposition 3.1 we calculate the probability at time  $t < \tau$  of default before  $T > t$  as

$$P[\tau \leq T | \mathcal{G}_t] = 1 - e^{A_t - A_T} = 1 - e^{-\int_t^T \lambda_s ds},$$

where the second equality follows from the existence of an intensity  $\lambda$ . We can



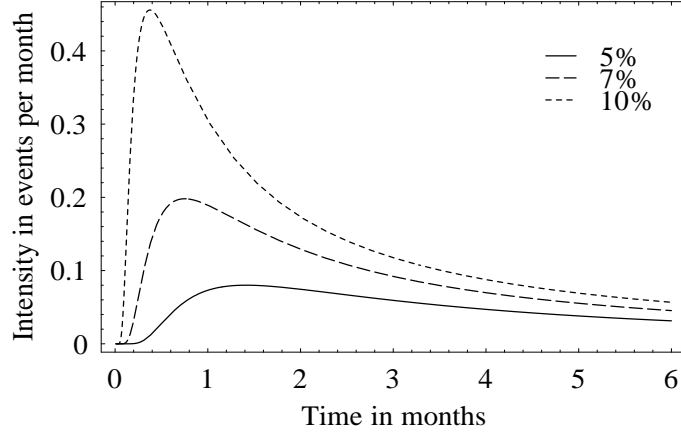


Figure 5: Default arrival intensity with observable default threshold, varying asset volatility.

also calculate the default probability directly:

$$\begin{aligned}
 P[\tau \leq T | \mathcal{G}_t] &= \frac{P[t < \tau \leq T]}{P[\tau > t]} = \frac{P[M_T \leq D] - P[M_t \leq D]}{1 - P[M_t \leq D]} \\
 &= \frac{\int_{-\infty}^0 (\Psi(T, x) - \Psi(t, x)) e^x dx}{1 - \int_{-\infty}^0 \Psi(t, x) e^x dx}. \quad (19)
 \end{aligned}$$

Noting that  $A_t = -\ln L_t$  for the continuous decreasing survival process  $L$  given by (11), the equivalence between these characterizations is obvious:

$$1 - e^{A_t - A_T} = 1 - \frac{L_T}{L_t} = \frac{P[t < \tau \leq T]}{P[\tau > t]}.$$

Also note that the intensity (17) obtained from our compensator characterization can be derived directly from (19) by calculating  $\lim_{h \downarrow 0} \frac{1}{h} P[\tau \leq t + h | \mathcal{G}_t]$ . This suggests to interpret  $\lambda$  as a conditional default arrival rate.

Assuming zero recovery and that the default is independent of risk-free interest rates, with default probabilities credit spreads can be calculated using (8). We plot the term structure of credit spreads in Figure 6, where we vary the asset volatility. Spreads are bounded away from zero for all maturities: incomplete information on threshold and assets implies spreads properties consistent with empirical observations. Short spreads  $S(t, T)$  for  $T \downarrow t$  are given by the intensity  $\lambda_t$ , cf. Theorem 3.3. Compare to the spreads with perfect information in Figure 1.

From (19), for  $D \in \mathcal{G}_0$  we obtain the conditional default probability ex-

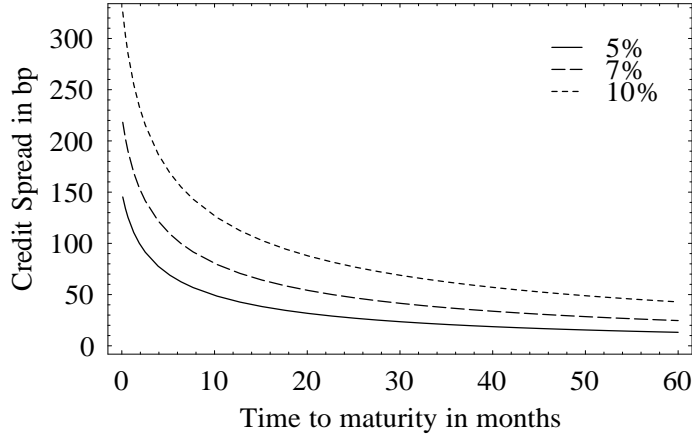


Figure 6: Term structure of credit spreads, varying asset volatility.

plicitly in terms of the normal distribution function:

$$P[\tau \leq T | \mathcal{G}_t] = \frac{\Psi(T, D) - \Psi(t, D)}{1 - \Psi(t, D)}, \quad t < \tau, \quad t < T.$$

For comparison, Figure 7 displays the term structure of credit spreads in that case (we again set  $D = -0.1$ ). As already verified by Duffie & Lando (2001), incomplete asset information is sufficient for the 'generation' of strictly positive spreads. The implied term structure shapes are somewhat different to those obtained with additionally imperfect threshold information. A variation in  $D$  has a qualitatively similar effect on spreads as a variation in  $\sigma$ .

## A Proofs

We start by proving the following two general results, both of which are needed to proof Propositions 3.1 and 3.2 as well as Theorem 3.3.

**Proposition A.1.** *Let the default stopping time  $\tau$  be totally inaccessible. Let, for a fixed time  $T$ ,  $Z$  be some bounded  $\mathcal{G}_T$ -measurable random variable. If the process  $Y$  defined by*

$$Y_t = E[Ze^{A_t - A_T} | \mathcal{G}_t], \quad t \leq T,$$

*is continuous at  $\tau$ , then on the set  $\{\tau > t\}$  we have a.s. that*

$$E[Z(1 - N_T) | \mathcal{G}_t] = E[Ze^{A_t - A_T} | \mathcal{G}_t], \quad t \leq T.$$

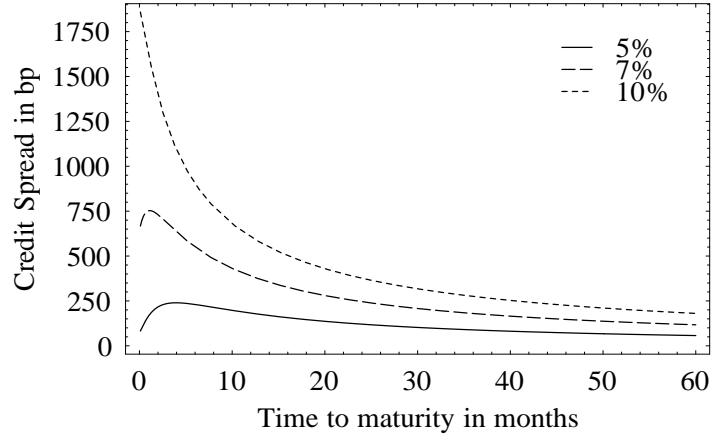


Figure 7: Term structure of credit spreads with observable default threshold, varying asset volatility.

PROOF OF PROPOSITION A.1. Letting  $K_t = E[Ze^{-A_t} | \mathcal{G}_t]$ , we can write  $Y_t = e^{A_t} K_t$ . Noting the continuity of  $A$ , by virtue of Itô's product rule we have

$$dY_t = e^{A_t} dK_t + Y_{t-} dA_t.$$

Denote by  $\Delta Z_t = Z_t - Z_{t-}$  the jump of the process  $Z$  at  $t$ . Defining  $U_t = (1 - N_t)Y_t$ , we find again with the aid of the product rule that

$$\begin{aligned} dU_t &= -Y_{t-} dN_t + (1 - N_{t-}) dY_t + \Delta(1 - N_t) \Delta Y_t \\ &= (1 - N_{t-}) e^{A_t} dK_t - Y_{t-} d(N_t - A_t) - N_{t-} Y_{t-} dA_t, \end{aligned} \quad (20)$$

where we have used our assumption that  $Y$  is continuous at  $\tau$  to set  $\Delta(1 - N_t) \Delta Y_t = 0$ . Now integration of both sides of (20) yields

$$U_T - U_t = \int_t^T (1 - N_{t-}) e^{A_t} dK_t - \int_t^T Y_{t-} d(N_t - A_t) - \int_t^T Y_{t-} N_{t-} dA_t. \quad (21)$$

Note that  $(K_t)_{0 \leq t \leq T}$  and  $N - A$  are martingales. Since the integrands are bounded and predictable, the first two terms of the right hand side of (21) are martingales. Using the fact that  $A$  is the compensator of  $N$ , we note that

$$\begin{aligned} E \left[ \int_t^T Y_{t-} N_{t-} dA_t \middle| \mathcal{G}_t \right] &= E \left[ \int_t^T Y_{t-} N_{t-} dN_t, \middle| \mathcal{G}_t \right] \\ &= E[1_{\{t < \tau \leq T\}} Y_{\tau-} N_{\tau-} | \mathcal{G}_t] \\ &= 0 \end{aligned}$$

because  $N_{\tau-} = 1_{\{\tau < \tau\}} = 0$ . Thus, taking conditional expectation of (21) yields

$$U_t = Y_t(1 - N_t) = E[U_T | \mathcal{G}_t] = E[Z(1 - N_T) | \mathcal{G}_t],$$

which is our assertion.  $\square$

**Proposition A.2.** *Let  $\tau$  be predictable with respect to  $(\mathcal{G}_t)$ . Defining  $Z_n := \{k2^{-n} \mid k = 0, 1, \dots\}$  for  $n \geq 1$ , on the set  $\{\tau > t\}$  we have  $P \times dt$  a.s. that*

$$\lim_{n \uparrow \infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} P[\tau \leq t_{i+1} | \mathcal{G}_{t_i}] 1_{\{t_i < t \leq t_{i+1}\}} = 0.$$

PROOF. Define the non-negative supermartingale  $M$  by  $M_t := 1 - N_t = 1_{\{\tau > t\}}$ . To  $M$  corresponds a unique finite measure  $P^M$  on the  $\sigma$ -field  $\mathcal{P}$  of predictable sets in  $\Omega \times (0, \infty]$  such that

$$P^M[B \times (t, \infty)] = E[M_t 1_B], \quad t \geq 0, \quad B \in \mathcal{G}_t,$$

cf. Föllmer (1972). Define furthermore a measure  $P^\lambda$  on  $\mathcal{P}$  such that

$$P^\lambda[B \times (t, T]] = (T - t)P[B], \quad 0 \leq t \leq T, \quad B \in \mathcal{G}_t.$$

Note that on  $\mathcal{P}$  the measure  $P^M$  has support  $S := \{(\omega, t) \mid \tau(\omega) = t\}$ . Since

$$P^\lambda[S] = \int P[d\omega] \int_0^\infty dt 1_B(\omega, t) = 0,$$

the measures  $P^M$  and  $P^\lambda$  are singular on  $\mathcal{P}$ . In a general semi-martingale setting, Airault & Föllmer (1974) introduced the Radon-Nikodym density  $dP^M/dP^\lambda$  of the absolutely continuous part of  $P^M$  with respect to  $P^\lambda$ . Now we have

$$\left. \frac{dP^M}{dP^\lambda} \right|_{\mathcal{P}} = 0.$$

The general results of Airault & Föllmer (1974) imply that the predictable density  $dP^M/dP^\lambda$  can be identified as

$$\frac{dP^M}{dP^\lambda}(\omega, t) = \lim_{n \uparrow \infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} E[M_{t_i} - M_{t_{i+1}} | \mathcal{G}_{t_i}](\omega) 1_{\{t_i < t \leq t_{i+1}\}} \quad P^\lambda - \text{a.s.}$$

Since  $E[M_{t_i} - M_{t_{i+1}} | \mathcal{G}_{t_i}] = P[\tau \leq t_{i+1} | \mathcal{G}_{t_i}] 1_{\{\tau > t_i\}}$ , our claim is proved.  $\square$

PROOF OF PROPOSITION 3.1. Follows from Proposition A.1 for  $Z = 1$ .  $\square$

PROOF OF PROPOSITION 3.2. Follows directly from Proposition A.1 by setting  $Z = X e^{-\int_t^T r_s ds}$  for  $X \in \mathcal{G}_T$ .  $\square$

PROOF OF THEOREM 3.3. (1) That the spread is non-negative follows trivially from its definition (8). If  $\tau$  is totally inaccessible, then we can apply Proposition A.1 to (9) to see that on the set  $\{\tau > t\}$  a.s.

$$\lim_{T \downarrow t} S(t, T) = -\frac{\partial}{\partial T} E[e^{A_t - A_T} | \mathcal{G}_t] \Big|_{T=t}$$

By dominated convergence and the fact that  $A_t = \int_0^t \lambda_s ds$ , we have

$$\begin{aligned} \lim_{T \downarrow t} S(t, T) &= -E\left[\frac{\partial}{\partial T} e^{A_t - A_T} | \mathcal{G}_t\right] \Big|_{T=t} \\ &= E[\lambda_T e^{A_t - A_T} | \mathcal{G}_t] \Big|_{T=t} \\ &= \lambda_t, \end{aligned}$$

which completes the proof of statement (1).

Now consider statement (2). If  $\tau$  is predictable,  $N$  is predictable as well and its decomposition is trivial: the compensator of  $N$  is  $A = N$  itself.  $A$  is thus not absolutely continuous. Using the definition of the spread, on  $\{\tau > t\}$  we have a.s.

$$\begin{aligned} &\lim_{n \uparrow \infty} \sum_{t_i \in Z_n} S(t_i, t_{i+1}) 1_{\{t_i < t \leq t_{i+1}\}} \\ &= -\lim_{n \uparrow \infty} \sum_{t_i \in Z_n} \frac{1}{t_{i+1} - t_i} (\ln P[\tau > t_{i+1} | \mathcal{G}_{t_i}]) 1_{\{t_i < t \leq t_{i+1}\}} \\ &= \lim_{n \uparrow \infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} \frac{1}{P[\tau > t_{i+1} | \mathcal{G}_{t_i}]} \Big|_{t_{i+1}=t_i} P[\tau \leq t_{i+1} | \mathcal{G}_{t_i}] 1_{\{t_i < t \leq t_{i+1}\}} \\ &= \lim_{n \uparrow \infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} P[\tau \leq t_{i+1} | \mathcal{G}_{t_i}] 1_{\{t_i < t \leq t_{i+1}\}}, \end{aligned}$$

which is zero by Proposition A.2.  $\square$

**Lemma A.3.** *Let  $K$  denote the  $(\mathcal{A}_t)$ -compensator of the survival process  $L$ , defined in (10). For all bounded and  $(\mathcal{A}_t)$ -predictable processes  $Z$  we have*

$$E[(N_s - N_t) Z_\tau | \mathcal{A}_t] = E\left[\int_t^s Z_u dK_u \Big| \mathcal{A}_t\right], \quad s \geq t.$$

PROOF OF LEMMA A.3. Analogous to Lemma 2.4 in Rutkowski (2000).  $\square$

PROOF OF THEOREM 4.1.  $K$  is the  $(\mathcal{A}_t)$ -compensator of the supermartingale  $L$  and thus by definition  $(\mathcal{A}_t)$ -predictable, right-continuous, increasing, and has  $K_0 = 0$ . Hence  $A$  is right-continuous, increasing,  $(\mathcal{G}_t)$ -predictable ( $\mathcal{A}_t \subseteq \mathcal{G}_t$  for all  $t$  and  $(L_{t-})_{t \geq 0}$  is  $(\mathcal{A}_t)$ -predictable as well), and satisfies  $A_0 = 0$ . It remains to show that the process  $N - A$  is a  $(\mathcal{G}_t)$ -martingale. Letting  $Z_t = (1 - N_t)/L_t$ , for  $s \geq t$  we compute

$$\begin{aligned} E[N_s - N_t | \mathcal{G}_t] &= E[N_s - N_t | \mathcal{A}_t \vee \sigma(\tau \wedge t)] \\ &= \frac{1 - N_t}{P[\tau > t | \mathcal{A}_t]} E[(N_s - N_t)(1 - N_t) | \mathcal{A}_t] \\ &= Z_t E[N_s - N_t | \mathcal{A}_t] \\ &= Z_t E[L_t - L_s | \mathcal{A}_t], \end{aligned}$$

where the last line follows from the definition of  $L$ , cf (10). On the other hand

$$\begin{aligned} E[A_s - A_t | \mathcal{G}_t] &= (1 - N_t) E \left[ \int_t^{s \wedge \tau} \frac{dK_u}{L_{u-}} \middle| \mathcal{A}_t \vee \sigma(\tau \wedge t) \right] \\ &= Z_t E \left[ \int_t^{s \wedge \tau} \frac{dK_u}{L_{u-}} \middle| \mathcal{A}_t \right] \\ &= Z_t E \left[ (1 - N_s) \int_t^s \frac{dK_u}{L_{u-}} + (N_s - N_t) \int_t^{s \wedge \tau} \frac{dK_u}{L_{u-}} \middle| \mathcal{A}_t \right] \\ &= Z_t E \left[ L_s \int_t^s \frac{dK_u}{L_{u-}} + (N_s - N_t) \int_t^{s \wedge \tau} \frac{dK_u}{L_{u-}} \middle| \mathcal{A}_t \right] \quad (22) \end{aligned}$$

$$= Z_t E \left[ L_s \int_t^s \frac{dK_u}{L_{u-}} + \int_t^s \int_t^{u \wedge \tau} \frac{dK_v}{L_{v-}} d(M_u - L_u) \middle| \mathcal{A}_t \right] \quad (23)$$

$$= Z_t E \left[ L_s(A_s - A_t) - \int_t^s A_u dL_u + A_t(L_s - L_t) \middle| \mathcal{A}_t \right] \quad (24)$$

$$= Z_t E[K_s - K_t | \mathcal{A}_t] \quad (25)$$

$$= Z_t E[L_t - L_s | \mathcal{A}_t], \quad (26)$$

where (22) follows by iterated expectations, (23) follows by Lemma A.3 and the fact that  $K = M - L$  for some  $(\mathcal{A}_t)$ -martingale  $M$ , and (24) is due to the definition of  $A$ . For (25) we have used the fact that

$$\begin{aligned} \int_t^s A_u dL_u &= A_s L_s - A_t L_t - \int_t^s L_{u-} dA_u \\ &= A_s L_s - A_t L_t - K_s + K_t, \end{aligned}$$

by virtue of the product formula (note that  $K$  is a process of bounded variation, so that  $\langle A \rangle = 0$ ). (26) follows again from the fact that  $K = M - L$  for some  $(\mathcal{A}_t)$ -martingale  $M$ . We have thus verified that

$$E[A_s - A_t | \mathcal{G}_t] = E[N_s - N_t | \mathcal{G}_t], \quad s \geq t,$$

proving that  $N - A$  follows a  $(\mathcal{G}_t)$ -martingale.

Now consider the second statement. If  $L$  is continuous and decreasing, then its  $(\mathcal{A}_t)$ -compensator  $K$  is given by  $K = 1 - L$ . In this case the default compensator  $A$  is continuous as well and we can write

$$dL_{t \wedge \tau} = -L_{t \wedge \tau} dA_t,$$

leading to the integrated form  $L_{t \wedge \tau} = e^{-A_t}$ .  $\square$

PROOF OF THEOREM 4.4. Under the assumed conditions on  $H$ , on the set  $\{\tau > t\}$  we have a.s.

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} E[K_{t+h} - K_t | \mathcal{A}_t] &= \lim_{h \downarrow 0} \frac{1}{h} E[L_t - L_{t+h} | \mathcal{A}_t] \\ &= \lim_{h \downarrow 0} \frac{1}{h} (P[D < M_t | \mathcal{A}_t] - P[D < M_{t+h} | \mathcal{A}_t]) \end{aligned} \quad (27)$$

$$\begin{aligned} &= \lim_{h \downarrow 0} \frac{1}{h} E[H(t+h, D) - H(t, D) | \mathcal{A}_t] \\ &= \lim_{h \downarrow 0} \int_{-\infty}^0 \frac{1}{h} E[H(t+h, x) - H(t, x) | \mathcal{A}_t] g(x) dx \end{aligned} \quad (28)$$

$$= \int_{-\infty}^0 \dot{H}(t, x) g(x) dx, \quad (29)$$

by dominated convergence. (27) is due to the definition of  $L$ . (28) follows from the definition of  $H(t, \cdot)$  as the  $\mathcal{A}_t$ -conditional distribution function of  $M_t$ , and the independence of  $D$  from  $\mathcal{A}_t$ . Since Aven's (1985) conditions are satisfied,  $K$  is absolutely continuous with respect to Lebesgue measure with a density given by (29). Our claim is then implied by Theorem 4.1, taking into account (11).  $\square$

PROOF OF PROPOSITION 4.5. The proof is analogous to that of Theorem 4.4. Under the assumed conditions on  $H$ , on the set  $\{\tau > t\} = \{M_t > D\}$  we have

a.s.

$$\begin{aligned}
& \frac{1}{L_t} \lim_{h \downarrow 0} \frac{1}{h} E[K_{t+h} - K_t \mid \mathcal{A}_t] \\
&= \lim_{h \downarrow 0} \frac{1}{h} P[M_{t+h} \leq D \mid \mathcal{A}_t, M_t > D] \\
&= \lim_{h \downarrow 0} \frac{1}{h} \int_{-\infty}^0 P[M_{t+h} \leq x \mid \mathcal{A}_t, M_t > D := x] P[D \in dx \mid \mathcal{G}_t] dx \\
&= \lim_{h \downarrow 0} \int_{-\infty}^0 \frac{\frac{1}{h} E[H(t+h, x) - H(t, x) \mid \mathcal{A}_t]}{1 - H(t, x)} g_t(x) dx \\
&= \int_{-\infty}^0 m_t(x) g_t(x) dx,
\end{aligned} \tag{30}$$

by dominated convergence. (30) follows from Bayes' rule. Since Aven's (1985) conditions are satisfied, our claim is implied by Theorem 4.1.  $\square$

**PROOF OF PROPOSITION 4.6.** We show that  $m_t(x) = \frac{1}{2}\sigma^2 a_z(t, x, x)$  for all  $x \leq 0$  and  $t < \tau$ . First observe that

$$\begin{aligned}
m_t(x) &= \lim_{h \downarrow 0} \frac{\frac{1}{h} E[H(t+h, x) - H(t, x) \mid \mathcal{A}_t]}{1 - H(t, x)} \\
&= \lim_{h \downarrow 0} \frac{\frac{1}{h} P[M_{t+h} \leq x < M_t \mid \mathcal{A}_t]}{P[x < M_t \mid \mathcal{A}_t]} \\
&= \lim_{h \downarrow 0} \frac{1}{h} P[M_{t+h} \leq x \mid \mathcal{A}_t, M_t > x].
\end{aligned}$$

On the set  $\{\tau > t\}$ , from the Markov property of Brownian motion and by substituting  $y = x - z/\sigma\sqrt{h}$  we obtain

$$\begin{aligned}
m_t(x) &= \lim_{h \downarrow 0} \frac{1}{h} \int_x^\infty P[M_h \leq x - z] a(t, x, z) dz \\
&= \sigma \lim_{h \downarrow 0} \int_{-\infty}^0 P[M_h \leq y\sigma\sqrt{h}] \frac{1}{\sqrt{h}} a(t, x, x - y\sigma\sqrt{h}) dy.
\end{aligned}$$

The probability  $P[M_t \leq x] = \Psi(t, x)$  is explicitly given in (15) and we get

$$\begin{aligned}
& \lim_{h \downarrow 0} P[M_h \leq y\sigma\sqrt{h}] \\
&= \lim_{h \downarrow 0} \left( 1 - \Phi\left(\frac{\mu\sqrt{h}}{\sigma} - y\right) + \exp\left(\frac{2\mu y\sqrt{h}}{\sigma}\right) \Phi\left(y + \frac{\mu\sqrt{h}}{\sigma}\right) \right) \\
&= 2\Phi(y),
\end{aligned}$$



where  $\Phi$  is the standard normal distribution function. Now, since  $a(t, x, x) = 0$ , we obtain

$$\lim_{h \downarrow 0} \frac{1}{\sqrt{h}} a(t, x, x - y\sigma\sqrt{h}) = y\sigma a_z(t, x, x),$$

where the derivative is taken from the right. By dominated convergence,

$$\begin{aligned} m_t(x) &= 2\sigma^2 a_z(t, x, x) \int_{-\infty}^0 \Phi(y) y dy \\ &= \frac{1}{2} \sigma^2 a_z(t, x, x). \end{aligned}$$

To justify dominated convergence, note that for  $h < 1$

$$|P[M_h \leq y\sigma\sqrt{h}]| \leq M(y) = 1 - \Phi(-y) + \exp(2|\mu|y\sigma^{-1})\Phi(y - |\mu|\sigma^{-1})$$

and  $M(y)$  goes to zero exponentially fast as  $y \rightarrow -\infty$ . Since  $a_z(t, x, z)$  is bounded, we have for some constant  $B$  that

$$|P[M_h \leq y\sigma\sqrt{h}] \frac{1}{\sqrt{h}} a(t, x, x - y\sigma\sqrt{h})| < M(y) B,$$

providing an integrable upper bound for all  $h < 1$ . □

## B The Compensator and a Change of Measure

If our risk-neutrality assumption is relaxed, the valuation of contingent claims by the no-arbitrage argument requires a change of the objective probability measure  $P$  to some equivalent measure  $\tilde{P} \approx P$  under which all discounted price processes are martingales (Harrison & Kreps (1979), Harrison & Pliska (1981)). To apply Proposition 3.2 in this case, we need the compensator  $\tilde{A}$  of  $\tau$  with respect to  $\tilde{P}$ . We now provide a characterization of  $\tilde{A}$  in terms of the  $P$ -compensator  $A$ . Our result is analogous to that of Artzner & Delbaen (1995, Appendix A1), who examine an intensity under a change of measure. If a  $P$ -intensity does in fact exist, both results are essentially equivalent.

**Proposition B.1.** *Let  $\tilde{P}$  be some probability measure equivalent to  $P$  and set  $Z = d\tilde{P}/dP$ . Define a martingale  $(Z_t)_{t \geq 0}$  by  $Z_t = E[Z | \mathcal{G}_t]$  and denote by  $K_\tau = E[Z | \mathcal{G}_{\tau-}]$  the predictable projection of  $Z$ . If  $\tau$  is totally inaccessible, then the  $\tilde{P}$ -compensator  $\tilde{A}$  can in terms of the  $P$ -compensator  $A$  be written as*

$$\tilde{A}_t = \int_0^t \frac{K_s}{Z_{s-}} dA_s, \quad t \geq 0.$$

PROOF. Clearly,  $\tilde{A}$  is increasing, predictable, and satisfies  $\tilde{A}_0 = 0$ . If the process  $N - \tilde{A}$  is a  $\tilde{P}$ -martingale, then  $\tilde{A}$  is the  $\tilde{P}$ -compensator of  $\tau$ . For all non-negative and predictable process  $C$  we have

$$\tilde{E}[C_\tau] = E[ZC_\tau] = E[E[Z | \mathcal{G}_{\tau-}]C_\tau] = E[K_\tau C_\tau].$$

Since  $KC$  is predictable and  $A$  is the  $P$ -compensator of  $\tau$  ( $N - A$  is a  $P$ -martingale and thus also  $(\int_0^t L_s d(N_s - A_s))_{t \geq 0}$  for all non-negative and predictable  $L$ ), we get

$$E[K_\tau C_\tau] = E \left[ \int_0^\infty K_t C_t dN_t \right] = E \left[ \int_0^\infty K_t C_t dA_t \right].$$

By Fubini's Theorem and the definition of  $Z_t$ ,

$$E \left[ \int_0^\infty K_t C_t dA_t \right] = \int_0^\infty E[K_t C_t] dA_t = \int_0^\infty \tilde{E} \left[ \frac{K_t C_t}{Z_t} \right] dA_t.$$

Now choose a càdlàg -version of the martingale  $(Z_t)_{t \geq 0}$ ; hence  $Z_t(\omega) = Z_{t-}(\omega)$ . Another application of Fubini's Theorem and the definition of  $\tilde{A}$  then yields

$$\int_0^\infty \tilde{E} \left[ \frac{K_t C_t}{Z_t} \right] dA_t = \tilde{E} \left[ \int_0^\infty \frac{K_t C_t}{Z_{t-}} dA_t \right] = \tilde{E} \left[ \int_0^\infty C_t d\tilde{A}_t \right].$$

We have thus shown that for all predictable  $C$

$$\tilde{E}[C_\tau] = \tilde{E} \left[ \int_0^\infty C_t dN_t \right] = \tilde{E} \left[ \int_0^\infty C_t d\tilde{A}_t \right],$$

implying that the process  $N - \tilde{A}$  is a  $\tilde{P}$ -martingale.  $\square$

Note that if a predictable  $P$ -intensity  $\lambda$  of  $\tau$  exists, then the  $\tilde{P}$ -intensity  $\tilde{\lambda}$  exists as well and is predictable. By virtue of Proposition B.1 we then have

$$\tilde{\lambda}_t = \frac{K_t}{Z_{t-}} \lambda_t,$$

which is the result of Artzner & Delbaen (1995).

## References

- Airault, Hélène & Hans Föllmer (1974), 'Relative densities of semimartingales', *Inventiones math.* **27**, 299–327.
- Anderson, Ronald & Suresh Sundaresan (1996), 'Design and valuation of debt contracts', *Review of Financial Studies* **9**, 37–68.

- Artzner, Philippe & Freddy Delbaen (1995), ‘Default risk insurance and incomplete markets’, *Mathematical Finance* **5**, 187–195.
- Aven, Terje (1985), ‘A theorem for determining the compensator of a counting process’, *Scandinavian Journal of Statistics* **12**, 69–72.
- Bélanger, Alain, Steven E. Shreve & Dennis Wong (2001), A unified model for credit derivatives. Working Paper, Department of Mathematics, Carnegie Mellon University.
- Beneish, M. & E. Press (1995), ‘Interrelation among events of default’, *Contemporary Accounting Research* **12**, 299–327.
- Black, Fischer & John C. Cox (1976), ‘Valuing corporate securities: Some effects of bond indenture provisions’, *Journal of Finance* **31**, 351–367.
- Black, Fischer & Myron Scholes (1973), ‘The pricing of options and corporate liabilities’, *Journal of Political Economy* **81**, 81–98.
- Borodin, A. & P. Salminen (1996), *Handbook of Brownian Motion: Facts and Formulae*, Birkhäuser, Basel.
- Brémaud, Pierre (1980), *Point Processes and Queues – Martingale Dynamics*, Springer-Verlag, New York.
- Duffie, Darrell & David Lando (2001), ‘Term structures of credit spreads with incomplete accounting information’, *Econometrica* **69**(3), 633–664.
- Duffie, Darrell & Kenneth J. Singleton (1999), ‘Modeling term structures of defaultable bonds’, *Review of Financial Studies* **12**, 687–720.
- Duffie, Darrell, Lasse Heje Pedersen & Kenneth J. Singleton (2000), Modeling sovereign yield spreads: A case study of Russian debt. Working Paper, GSB, Stanford University.
- Duffie, Darrell, Mark Schroder & Costis Skiadas (1996), ‘Recursive valuation of defaultable securities and the timing of resolution of uncertainty’, *Annals of Applied Probability* **6**, 1075–1090.
- El Karoui, Nicole & Lionel Martellini (2001), A theoretical inspection of the market price for default risk. Working Paper, Marshall School of Business, University of Southern California.
- Elliott, Robert, Monique Jeanblanc & Marc Yor (2000), ‘On models of default risk’, *Mathematical Finance* **10**(2), 179–195.
- Föllmer, Hans (1972), ‘The exit measure of a supermartingale’, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **21**, 154–166.

- Föllmer, Hans & Martin Schweizer (1990), Hedging of contingent claims under incomplete information, *in* M.Davis & R.Elliot, eds, ‘Applied Stochastic Analysis’, Gordon and Breach, London, pp. 389–414.
- Geske, Robert (1979), ‘The valuation of compound options’, *Journal of Financial Economics* **7**, 63–81.
- Harrison, J. Michael & David Kreps (1979), ‘Martingales and arbitrage in multi-period securities markets’, *Journal of Economic Theory* **20**, 381–408.
- Harrison, J. Michael & Stanley R. Pliska (1981), ‘Martingales and stochastic integrals in the theory of continuous trading’, *Stochastic Processes and Their Applications* **11**, 215–260.
- Jarrow, Robert A. & Stuart M. Turnbull (1995), ‘Pricing derivatives on financial securities subject to credit risk’, *Journal of Finance* **50**(1), 53–86.
- Jeanblanc, Monique & Marek Rutkowski (1999), Modeling of default risk: An overview. Working Paper, Department of Mathematics, Université d’Evry.
- Kusuoka, Shigeo (1999), ‘A remark on default risk models’, *Advances in Mathematical Economics* **1**, 69–82.
- Lando, David (1998), ‘On cox processes and credit risky securities’, *Review of Derivatives Research* **2**, 99–120.
- Leland, Hayne E. (1994), ‘Corporate debt value, bond covenants, and optimal capital structure’, *Journal of Finance* **49**(4), 1213–1252.
- Leland, Hayne E. & Klaus Bjerre Toft (1996), ‘Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads’, *Journal of Finance* **51**(3), 987–1019.
- Longstaff, Francis A. & Eduardo S. Schwartz (1995), ‘A simple approach to valuing risky fixed and floating rate debt’, *Journal of Finance* **50**(3), 789–819.
- Merton, Robert C. (1974), ‘On the pricing of corporate debt: The risk structure of interest rates’, *Journal of Finance* **29**, 449–470.
- Rutkowski, Marek (2000), On models of default risk. Working Paper, Department of Mathematics, University of Warsaw.
- Sarig, Oded & Arthur Warga (1989), ‘Some empirical estimates of the risk structure of interest rates’, *Journal of Finance* **44**, 1351–1360.
- Yor, Marc (1994), Local times and excursions for brownian motion, a concise introduction. Facultad de Ciencias, Universidad Central de Venezuela.